

Multipole field calculation for beam tracking

The general formula for multipole potential is

$$\varphi_n = \frac{B_n}{n \cdot a^{(n-1)}} \cdot r^n \sin(n \cdot \theta), \quad (1)$$

where n=1 is for dipole, n = 2 for quadrupole, and so on.

Quadrupole:

The quadrupole potential in Cartesian coordinate can be written as:

$$\varphi_2 = \frac{B_2}{2 \cdot a} \cdot r^2 \sin(2 \cdot \theta) = \frac{B_2}{a} \cdot xy$$

From this we derive the magnetic field:

$$\begin{aligned} B_x &= \frac{B_2}{a} \cdot y \\ B_y &= \frac{B_2}{a} \cdot x \end{aligned} \quad (2a)$$

Skew quadrupole:

The skew quadrupole potential in Cartesian coordinate can be written as:

$$\begin{aligned} \psi_2 &= \frac{A_2}{2 \cdot a} \cdot r^2 \sin\left[2 \cdot \left(\theta - \frac{\pi}{4}\right)\right] \\ &= \frac{-A_2}{2 \cdot a} \cdot r^2 \cos(2\theta) \\ &= \frac{-A_2}{2 \cdot a} \cdot r^2 (\cos^2 \theta - \sin^2 \theta) \\ &= \frac{-A_2}{2a} \cdot (x^2 - y^2) \end{aligned}$$

From this we derive the magnetic field:

$$\begin{aligned} B_x &= \frac{A_2}{a} \cdot (-x) \\ B_y &= \frac{A_2}{a} \cdot y \end{aligned} \quad (2b)$$

Sextupole:

The sextupole potential in Cartesian coordinate can be written as:

$$\begin{aligned}\varphi_3 &= \frac{B_3}{3 \cdot a^2} \cdot r^3 \sin(3 \cdot \theta) \\ &= \frac{B_3}{3 \cdot a^2} \cdot r^3 \cdot (3 \cos^2 \theta \cdot \sin \theta - \sin^3 \theta) \\ &= \frac{B_3}{3a^2} [3x^2y - y^3]\end{aligned}$$

From this we derive the magnetic field:

$$\begin{aligned}B_x &= \frac{B_3}{a^2} \cdot [2xy] \\ B_y &= \frac{B_3}{a^2} \cdot [x^2 - y^2]\end{aligned}\tag{3a}$$

Skew sextupole:

The skew sextupole potential in Cartesian coordinate can be written as:

$$\begin{aligned}\psi_3 &= \frac{A_3}{3 \cdot a^2} \cdot r^3 \sin\left[3 \cdot \left(\theta - \frac{\pi}{6}\right)\right] \\ &= \frac{-A_3}{3 \cdot a} \cdot r^3 \cos(3\theta) \\ &= \frac{-A_3}{3 \cdot a^2} \cdot r^3 (\cos^3 \theta - 3 \cos \theta \cdot \sin^2 \theta) \\ &= \frac{-A_3}{3a^2} (x^3 - 3xy^2)\end{aligned}$$

From this we derive the magnetic field:

$$\begin{aligned}B_x &= \frac{-A_3}{a^2} \cdot (x^2 - y^2) \\ B_y &= \frac{A_3}{a^2} \cdot (2xy)\end{aligned}\tag{3b}$$

Octupole:

The octupole potential in Cartesian coordinate can be written as:

$$\varphi_4 = \frac{B_4}{4 \cdot a^3} \cdot r^4 \sin(4 \cdot \theta) = \frac{B_4}{a^3} [x^3y - xy^3]$$

From this we derive the magnetic field:

$$\begin{aligned}B_x &= \frac{B_4}{a^3} \cdot [3x^2y - y^3] \\ B_y &= \frac{B_4}{a^3} \cdot [x^3 - 3xy^2]\end{aligned}\tag{4a}$$

Skew octupole:

The skew octupole potential in Cartesian coordinate can be written as:

$$\begin{aligned}\psi_4 &= \frac{-A_4}{4 \cdot a^3} \cdot r^4 \cos(4 \cdot \theta) \\ &= \frac{-A_4}{4 \cdot a^3} \cdot r^4 \cdot (\cos^4 \theta - 6\sin^2 \theta \cdot \cos^2 \theta + \sin^4 \theta) \\ &= \frac{-A_4}{4 \cdot a^3} [x^4 - 6x^2y^2 + y^4]\end{aligned}$$

From this we derive the magnetic field:

$$\begin{aligned}B_x &= \frac{-A_4}{a^3} \cdot [x^3 - 3xy^2] \\ B_y &= \frac{A_4}{a^3} \cdot [3x^2y - y^3]\end{aligned}\tag{4b}$$

Decapole:

The decapole potential in Cartesian coordinate can be written as:

$$\begin{aligned}\varphi_5 &= \frac{B_5}{5 \cdot a^4} \cdot r^5 \sin(5 \cdot \theta) \\ &= \frac{B_5}{5 \cdot a^4} \cdot r^5 \cdot (5\cos^4 \theta \cdot \sin \theta - 10\cos^2 \theta \cdot \sin^3 \theta + \sin^5 \theta) \\ &= \frac{B_5}{5a^4} [5x^4y - 10x^2y^3 + y^5]\end{aligned}$$

From this we derive the magnetic field:

$$\begin{aligned}B_x &= \frac{B_5}{a^4} \cdot [4x^3y - 4xy^3] \\ B_y &= \frac{B_5}{a^4} \cdot [x^4 - 6x^2y^2 + y^4]\end{aligned}\tag{5a}$$

Skew decapole:

The skew decapole potential in Cartesian coordinate can be written as:

$$\begin{aligned}\psi_5 &= \frac{-A_5}{5 \cdot a^4} \cdot r^5 \cos(5 \cdot \theta) \\ &= \frac{-A_5}{5 \cdot a^4} \cdot r^5 \cdot (\cos^5 \theta - 10\cos^3 \theta \cdot \sin^2 \theta + 5\cos \theta \cdot \sin^4 \theta) \\ &= \frac{-A_5}{5a^4} [x^5 - 10x^3y^2 + 5xy^4]\end{aligned}$$

From this we derive the magnetic field:

$$\begin{aligned}B_x &= \frac{-A_5}{a^4} \cdot [x^4 - 6x^2y^2 + y^4] \\ B_y &= \frac{A_5}{a^4} \cdot [4x^3y - 4xy^3]\end{aligned}\tag{5b}$$

Rolled multipoles

Rolling the multipole by an angle $+\varepsilon$ such that

$$\theta = \bar{\theta} + \varepsilon$$

and following equation (1) the vector potential of the n-th multipole can be written as:

$$\varphi_n = \frac{B_n}{n \cdot a^{(n-1)}} \cdot \bar{r}^n \sin(n\bar{\theta}), \text{ with } \bar{r} = r.$$

In the original frame of reference this will be:

$$\begin{aligned} \varphi_n &= \frac{B_n}{n \cdot a^{(n-1)}} \cdot r^n \sin[n \cdot (\theta - \varepsilon)] \\ &= \frac{B_n}{n \cdot a^{(n-1)}} \cdot r^n \sin(n\theta - n\varepsilon) \\ &= \frac{B_n}{n \cdot a^{(n-1)}} \cdot r^n \cdot [\sin(n\theta)\cos(n\varepsilon) - \cos(n\theta)\sin(n\varepsilon)] \\ &= \frac{B_n \cos(n\varepsilon)}{n \cdot a^{(n-1)}} \cdot r^n \sin(n\theta) - \frac{B_n \sin(n\varepsilon)}{n \cdot a^{(n-1)}} \cdot r^n \cos(n\theta) \\ &= N_n r^n \sin(n\theta) + S_n r^n \cos(n\theta) \end{aligned} \tag{6}$$

where $N_n = \frac{B_n \cos(n\varepsilon)}{n \cdot a^{(n-1)}}$ is the n-th order normal component and $S_n = \frac{-B_n \sin(n\varepsilon)}{n \cdot a^{(n-1)}}$ is the n-th order skew component.

Given respective normal an skew components equation (6) can also be used to figure out the equivalent roll angle of a multipole field.